Renegotiation-proof Equilibrium

We will consider the case where $n=2$ to avoid complications in ~~[illegible]~~ definitions (as we would need to think about coallitions etc).

Let $\sigma=(\sigma_1, \sigma_2)$ be a SPE of the repeated game. Define:

$$S(\sigma) := \{(\sigma_1', \sigma_2') \mid (\sigma_1', \sigma_2') \text{ is a continuation equilibrium of } \sigma\}.$$

Definition (Weak renegotiation-proofness)

σ is Weak renegotiation-proof (WRP) if $\forall \sigma', \sigma'' \in S(\sigma)$, σ' does not Pareto dominate σ''.

"an internal consistency requirement"

Theorem (Folk-like theorem for renegotiation-proof equilibria)

Let $(v_1, v_2) \in V^*$. If $\exists (s_1^1, s_2^1)$ and (s_1^2, s_2^2) such that:

(1) $\max_{s_1} g_1(s_1, s_2^1) < v_1$ and $g_2(s_1^1, s_2^1) > v_2$ — player 1 can be punished

(2) $\max_{s_2} g_2(s_1^2, s_2) < v_2$ and $g_1(s_1^2, s_2^2) > v_1$ — player 2 can be punished

Then $\forall \delta$ near 1, there exists a WRP equilibrium with average payoffs (v_1, v_2) in the game with discount factor δ.

Conversely, if σ is a WRP equilibrium for δ with average payoffs (v_1, v_2), $\exists (s_1^1, s_2^1)$, (s_1^2, s_2^2) which satisfy (1) and (2) weakly.

Proof:

($\Rightarrow$) Such A & B must exist otherwise $(v_1, v_2) \notin V^*$.

If orange line had hit $g(s_1, s_2)$ we could randomize just the red points to get the average payoff (v_1, v_2).

Suppose that player 1 randomizes between (s_1^A, s_1^B) with probabilities $(p, 1-p)$. Suppose player 2 randomizes independently with probabilities (s_2^A, s_2^B), $(p, 1-p)$.

We get the blue path, which necessarily intersects the dotted line, so there is some p which gets us to this intersection.

Randomize between this and (s_1^1, s_2^1) to get the normal-phase payoff (v_1, v_2).

To complete the sufficiency proof, we need to describe the punishment phase.

Punish by playing (s_1', s_2'), players get $(v_1', v_2') = g(s_1', s_2')$.
If we punish for 2 periods, player 1 gets:

$$v_1' + \delta v_1' + \frac{\delta^2}{1-\delta} v_1$$

The continuation payoffs are the red X on the previous page and converge to (v_1, v_2).

($\Leftarrow$) Let σ be a WRP equilibrium for δ, with average payoffs (v_1, v_2).

Let σ' be the continuation equilibrium with the <u>lowest</u> payoff for 1.
- if multiple equilibria exist, choose the one with highest payoff for player 2.

Let $g_1^*(\sigma')$ be player 1's average payoff in the repeated game if players are playing σ'. Then $g_1^*(\sigma') \leq v_1$.

<u>Claim:</u> $g_2^*(\sigma') \geq v_2$.
To show this note that if $g_1^*(\sigma') < v_1$, follows from definition of WRP.
if $g_1^*(\sigma') = v_1$, then follows from σ'.

Let (s_1', s_2') be the first period actions of σ', and let $\hat{\sigma}'$ be the continuation equilibrium of σ'.

<u>Claim:</u> $\max_{s_1} g_1(s_1, s_2') \leq v_1$ and $g_2(s_1', s_2') \geq v_2$.

If $g_2(s_1', s_2') < g_2^*(\sigma')$ then $g_2^*(\hat{\sigma}') > g_2^*(\sigma') \Rightarrow \underline{g_1^*(\hat{\sigma}') \leq g_1^*(\sigma')}$
this is a contradiction by definition of σ'.

Thus $g_2(s_1', s_2') \geq g_2^*(\sigma') \geq v_2$.

Now assume by way of contradiction $\max_{s_1} g_1(s_1, s_2') > g_1^*(\sigma')$.
Then player 1 can deviate in 1st period of σ' to get a higher payoff than $g_1^*(\sigma')$ $\Rightarrow\Leftarrow$ (contradicts that σ' is an equilibrium).

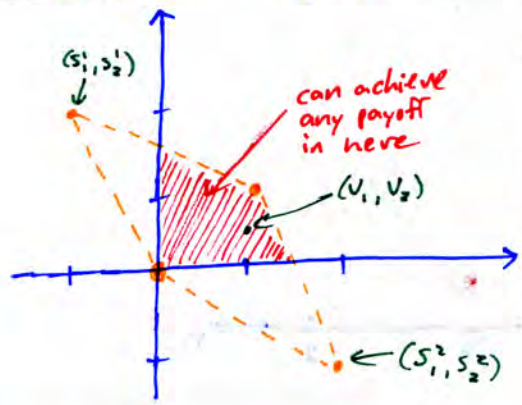
Thus: $\max_{s_1} g(s_1, s_2') \leq g_1^*(\sigma') \leq v_1$. ∎

ECO512
Eric Maskin

Week 3.2

EXAMPLE — Prisoners' Dilemma

	C	D
c	1, 1	−1, 2
d	2, −1	0, 0

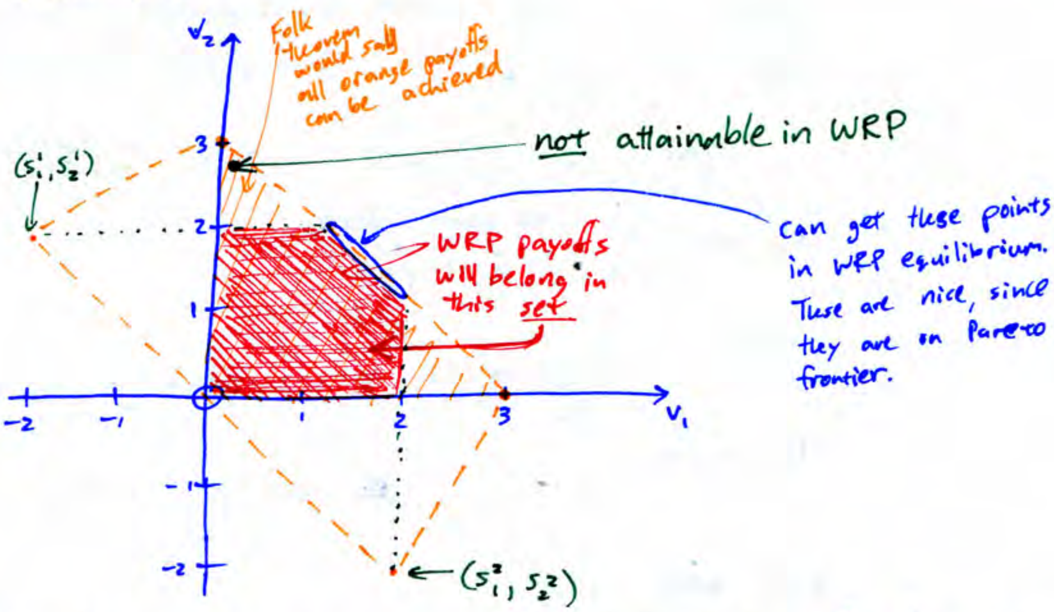


The requirement that an equilibrium is WRP does not constrain the equilibrium payoffs in this game.

EXAMPLE — Adjusted Prisoners' Dilemma (Advertising Game) in Farrell–Maskin (1989)

	C	M	D
c	1, 1	3, 0	−2, 2
m	0, 3	$\frac{4}{3}, \frac{4}{3}$	0, 0
d	2, −2	0, 0	0, 0

minimax point (0,0).



Definition Let $W(\delta) = \{(v_1, v_2) : (v_1, v_2)$ is attainable in WRP equilibrium for discount factor $\delta\}$.

Remark It would be nice if $P(V^*) \cap W(\delta) \neq \emptyset$ when $\delta \to 1$.

$P(V^*)$: Pareto Frontier of V^*.

side comment [N.B. WRP equilibria always exist $\Rightarrow$ Play 1-shot NE repeatedly.]

However, not all games (like the two above) achieve the Pareto frontier if we are confined to payoffs supported by WRP equilibria, as the next example will show.

EXAMPLE

	R	S	P
P	1,0	0,1	0,0
R	0,0	1,0	0,1
S	0,1	0,0	1,0

Unique mixed strategy NE where players play $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.
Minimax payoff of $\frac{1}{3}$ for each player.

Claim: Unique WRP equilibrium payoffs for any $\delta < 1$ are $(\frac{1}{3}, \frac{1}{3})$.

Proof: Suppose (v_1, v_2) is a WRP equilibrium achieved by σ.
Let σ' be the worst continuation equilibrium for player 1.
Let (s_1', s_2') be the first period strategies of σ', then:

$$\max_{s_1} g_1(s_1, s_2') \leq v_1 \quad \text{and} \quad g_2(s_1', s_2') \geq v_2 .$$

Write:
$$s_2' = (\underset{\text{rock}}{p}, \underset{\text{scissors}}{q}, \underset{\text{paper}}{1-p-q})$$

Then:
$$\max_{s_1} g_1(s_1, s_2') = \max\{p, q, 1-p-q\}$$

But also:
$$g_2(s_1', s_2') \leq \max\{p, q, 1-p-q\}$$

But then
$$v_1 \geq \max_{s_1} g_1(s_1, s_2') \geq g_2(s_1', s_2') \geq v_2$$

Similarly $v_2 \geq v_2$ (repeating argument for player 2).
Thus we must have $v_1 = v_2$ in any WRP equilibrium.
Since any payoff higher than $(\frac{1}{3}, \frac{1}{3})$ would violate the pareto property (we would have to punish by $(\frac{1}{3}, \frac{1}{3})$), then $(\frac{1}{3}, \frac{1}{3})$ is the only WRP equilibrium.
This is because we cannot punish a player without using a Pareto dominated strategy.

Theorem (Evans-Maskin (1989) Games and Ec Behaviour)

In a generic finite game, $P(v^*) \cap W(\delta) \neq \emptyset$ for δ near 1.
2 player game

A generic finite game:
If player 1 has m_1 strategies and player 2 has m_2 strategies
For an open and dense set of $2m_1m_2$-dimensional vector
finite games
$P(v^*) \cap W(\delta) \neq \emptyset$ for δ near 1.

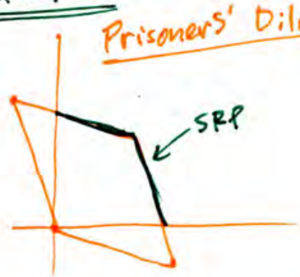
Proof: See paper.

Definition (Strongly renegotiation proof equilibrium)

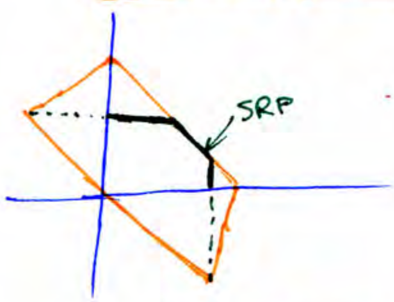
A WRP σ is Strongly renegotiation-proof (SRP) if there is no other WRP σ' that Pareto dominates σ.
"an external consistency requirement"

Examples

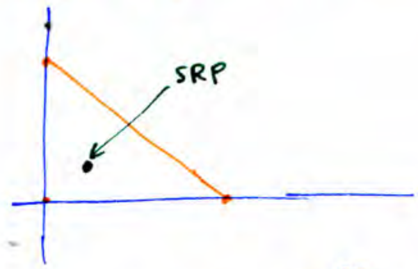
Prisoners' Dilemma



Advertising game



Rock-Paper-Scissors



Theorem For a generic finite ~~stage~~ game, there exist efficient SRP equilibria for δ near 1.

Next week: Imperfectly observed actions
Maskin-Fudenberg-Levine
Abreu-Pearce-Stacchetti

Repeated Games with Imperfect Monitoring

References: Abreu-Pearce-Stacchetti (1990) Econometrica
Fudenberg-Levine-Maskin (1994) Econometrica

Actions (incl. mixed strategies) of players are no longer observed ex-post.
Now, we ~~~~ only observe public outcome y.
Let $\pi(y|s_1,s_2)$ = Prob of outcome y given actions s_1 & s_2.

Note; We will consider 2 player games, but results hold for n-players

$$g_i(s_1,s_2) = \sum_y \pi(y|s_1,s_2)\, r_i(y,s_i) \leftarrow \text{expected payoff of player } i$$

where $r_i(y,s_i)$ is player i's payoff if i chooses s_i and public outcome is y.

EQUILIBRIUM CONCEPT:

Perfect public equilibrium – i's action in period t, s_i^t depends only on past public outcomes $(y^1,\dots,y^{t-1})$.

restrict players not to condition on their own past actions – this retains the recursive structure of the game

EXAMPLE (A Prisoners' dilemma – Variant)

Players can either work or shirk, $y = H$ or L
If $y=H$, then players get a payoff of 6 each
If $y=L$, then each player's gross payoff is 0.
If a player plays W (work) they have a disutility of -3. If a player Shirks, there is no disutility.

	W	S
W	H: 3,3 / L: -3,-3	H: 3,6 / L: -3,0
S	H: 6,3 / L: 0,-3	H: 0,0 / L: 0,0

If WW then $P(H)=\frac{2}{3}$, $P(L)=\frac{1}{3}$
WS then $P(H)=\frac{1}{3}$, $P(L)=\frac{2}{3}$
SS then $P(H)=0$, $P(L)=1$

in expectation $\longrightarrow$

	W	S
W	1,1	-1,2
S	2,-1	0,0

If perfect monitoring and δ close to 1, we could sustain WW (payoffs of (1,1)).
However, in this game $\nexists$ PPE in which avg. payoffs sum to more than 1, $\forall \delta < 1$.

Fix $\delta < 1$.

Proof: Let $\gamma = \max\{v_1 + v_2 \mid (v_1, v_2) \text{ corresponds to a PPE}\}$

Consider PPE (v_1, v_2) with $v_1 + v_2 = \gamma$.

Continuation payoff if outcome is high and low, respectively.

Then $$v_i = (1-\delta) g_i(\underbrace{s_1, s_2}_{\text{can be mixed strategies}}) + \delta\left[\pi(H \mid s) w_i(H) + \pi(L \mid s) w_i(L)\right] \quad (*)$$

with $s = (s_1, s_2)$.

Lemma: If $\gamma > 1$, s_1 and s_2 both put positive probability on [redacted] work.

If either put zero probability to work, then (w, w) is never achieved and the sum of payoffs would be 1 or less (see PD).

if μ_i = Prob. player i plays w in first period, equation $(*)$ becomes:

payoff from working $\geq$ payoff from shirking (else would not have positive prob. on work.)

$$V_1 = (1-\delta)(2\mu_2 - 1) + \delta\left[\left(\tfrac{2}{3}\mu_2 + \tfrac{1}{3}(1-\mu_2)\right) W_1(H) + \left(\tfrac{1}{3}\mu_2 + \tfrac{2}{3}(1-\mu_2)\right) W_1(L)\right]$$
$$\geq (1-\delta) 2\mu_2 + \delta\left[\tfrac{1}{3}\mu_2 W_1(H) + \left(1 - \tfrac{1}{3}\mu_2\right) W_1(L)\right]$$

or if $$\boxed{W_1(H) \geq \frac{3(1-\delta)}{\delta} + W_1(L)}$$

Thus: $$V_1 \leq (1-\delta)(2\mu_2 - 1) + \delta\left[\left(\tfrac{2}{3}\mu_2 + \tfrac{1}{3}(1-\mu_2)\right) W_1(H) + \left(\tfrac{1}{3}\mu_2 + \tfrac{2}{3}(1-\mu_2)\right) \frac{3(1-\delta)}{\delta} W(H)\right]$$
$$\cancel{= (1-\delta)(2\mu_2 - 1) + \delta \tfrac{1}{3} W_1(H) - (1-\delta) W_1(H)}$$
$$= (1-\delta)(2\mu_2 - 1) + \delta\left(\tfrac{1}{3} + \tfrac{1}{3}\mu_2\right) W_1(H) - (2-\mu_2)(1-\delta) \cancel{\quad} + \delta\left(\tfrac{2}{3} - \tfrac{1}{3}\right)\mu_2 W(H)$$
$$= \cancel{(1-\delta)(2\mu_2 - 1) + (\delta(} \; \delta\left(\tfrac{1}{3} + \tfrac{2}{3}\right) W_1(H)$$

$\boxed{V_1 \leq \delta W_1(H)}$, similarly $V_2 \leq \delta W_2(H)$

Thus $V_1 + V_2 \leq \delta(W_1(H) + W_2(H)) < W_1(H) + W_2(H)$ $\Rightarrow\Leftarrow$.

Therefore $\gamma \leq 1$. ∎

Need to differentiate between who is working and not (else we cannot punish efficiently). Both could be working in the above example but the outcome can be low — so we want to punish both, but this is inefficient.

We need to have another public outcome. Call this "medium."

→ We really want to distinguish the deviations of each player.

Repeated Games

EXAMPLE (Addition of another public outcome)

Probabilites of outcome given strategies:

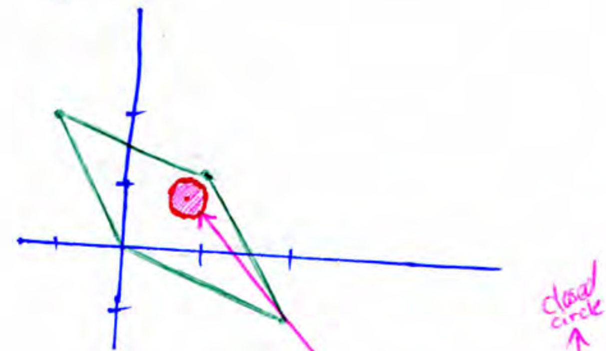
	H	M	L
w w	1/3	1/2	1/6
s w	1/3	0	2/3
w s	0	1/2	1/2
s s	0	0	1

Disutility of work = 3

H (6, 6)
M (4, 4)
L (0, 0)

in expectation

	w	s
w	1, 1	-1, 2
s	2, -1	0, 0



Claim: For all δ near 1 (for the same δ), all the points in the shaded circle, W, can be attained in PPE.

Sub-claim: For δ near 1, all points on the boundary of the ball (call these points (v_1, v_2)) can be decomposed so that there exist (s_1, s_2), such that s_1, s_2 are strategies in the stage game, and continuation payoffs in the ball: $w(H), w(M), w(L) \in W$ with:

W is self-decomposable if it satisfies this hypothesis for some δ.

$$V_i = (1-\delta) g_i(s_1, s_2) + \delta \sum_y \pi(y|s_1, s_2) w_i(y)$$
$$\geq (1-\delta) g_i(s_i', s_j) + \delta \sum_y \pi(y|s_1, s_2) w_i(y) \qquad \forall s_i'.$$

Proof of subclaim: (later)

Sub-Claim *: If for δ near 1, all points in W can be decomposed, then all points in W are attainable as PPE.

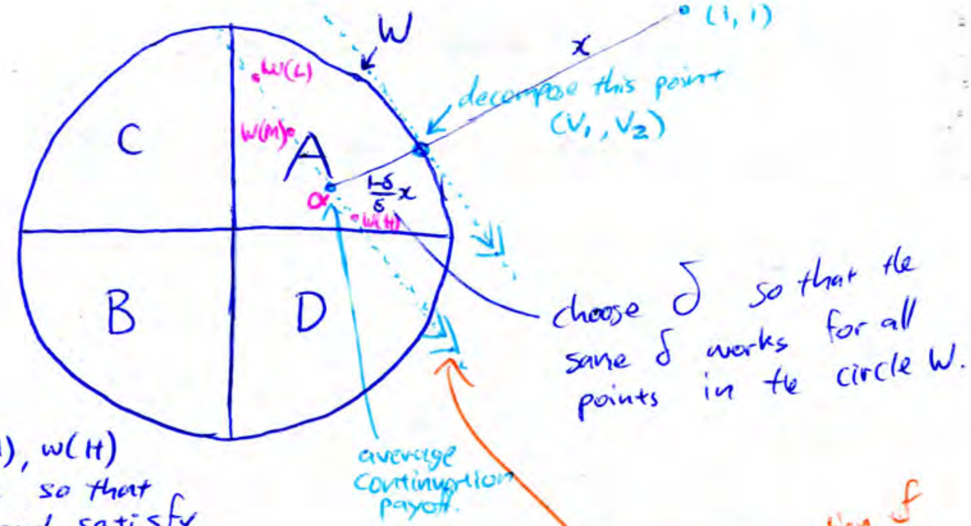
Proof of sub-claim*: $(v_1, v_2) W$ 1st ____ Continuation ____

Perform iterative sequence of decompositions
depends on whole history of outcomes.

$\Rightarrow$ we are left to show subclaim.

Proof of sub-claim:

Decomposing (v_1, v_2)
1st period actions (W, W), payoff $(1,1)$



assume equation of this line is:
$\beta_1 w_1 + \beta_2 w_2 = C$

$$\boxed{\Rightarrow w_2(y) = \frac{C - \beta_1 w_1(y)}{\beta_2}}$$

Sub-Sub Claim We can choose $w(L), w(M), w(H)$ on the second dotted line so that payoffs average out to α and satisfy the decomposition requirements and are interior to W.

$$v_1 \geq 2(1-\delta) + \delta\left(\tfrac{1}{3}w_1(H) + \tfrac{2}{3}w_1(L)\right)$$
$$v_2 \geq 2(1-\delta) + \delta\left(\tfrac{1}{2}w_2(M) + \tfrac{1}{2}w_2(L)\right)$$

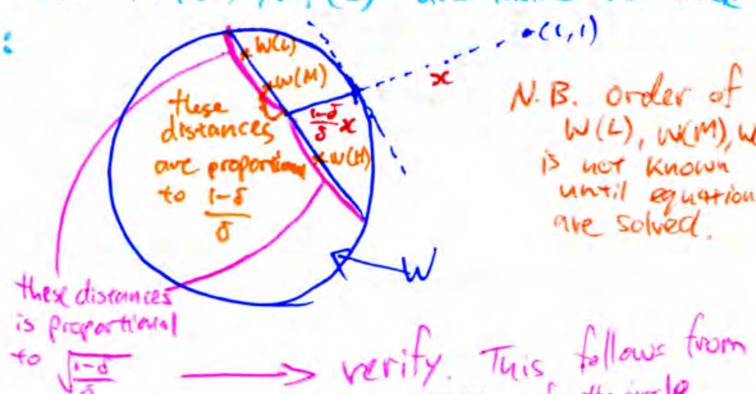
$$v_1 = (1-\delta) + \delta\left(\tfrac{1}{3}w_1(H) + \tfrac{1}{2}w_1(M) + \tfrac{1}{6}w_1(L)\right)$$

Substituting $w_2(y) = \frac{C - \beta_1 w_1(y)}{\beta_2}$, we get:

$$v_1 = 2(1-\delta) + \delta\left(\tfrac{1}{3}w_1(H) + \tfrac{2}{3}w_1(L)\right)$$
$$v_2 = 2(1-\delta) + \delta\left(\tfrac{1}{2}\left(\frac{C - \beta_1 w_1(M)}{\beta_2}\right) + \tfrac{1}{2}\left(\frac{C - \beta_1 w_1(L)}{\beta_2}\right)\right)$$
$$v_1 = (1-\delta) + \delta\left(\tfrac{1}{3}w_1(H) + \tfrac{1}{2}w_1(M) + \tfrac{1}{6}w_1(L)\right)$$

The resulting system has full rank since $\begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & \frac{1}{2}\left(\frac{-\beta_1}{\beta_2}\right) & \frac{1}{2}\left(\frac{-\beta_1}{\beta_2}\right) \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$ is full rank

Need to verify that the resulting $w_1(H), w_1(M), w_1(L)$ are inside the circle W. This is a matter of geometry:



N.B. order of $w(L), w(M), w(H)$ is not known until equations are solved.

⟶ verify. This follows from geometry of the circle.

As $\delta \to 1$, the ratio $\dfrac{\frac{1-\delta}{\delta}}{\sqrt{\frac{1-\delta}{\delta}}} \to 0$, so for some δ the continuation payoffs must lie within ~~W~~. ■

Repeated Games

How do we decompose the other quadrants?

For C use $(-1, 2)$, i.e (w, s) as first action.

For D use $(2, -1)$, i.e. (s, w) as first action.

For B use $(0,0)$, i.e. (s, s) as first action.

How does this relate to the number of outcomes?

Why did this decomposition not work in the 2 outcome case?

To create matric $\begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$ look at efficient actions (s_1^*, s_2^*)

get $\pi(s_1^*, s_2^*)$

Make sure player 1 does not have incentive to deviate by considering $\pi(s_1', s_2^*)$ and similarly for player 2 by looking at $\pi(s_1^*, s_2')$.

Thus if player 1 has m_1 actions and player 2 has m_2 actions we need $m_1 + m_2 - 1$ equations to have a full-ranked system.

$$\begin{pmatrix} \pi(s_1^*, s_2^*) \\ \pi(s_1', s_2^*) \\ \pi(s_1^*, s_2') \end{pmatrix}$$

Definition Game G has pairwise full rank if $\exists (s_1^*, s_2^*)$ such that the matrix has rank $m_1 + m_2 - 1$.

What if we have 3 players.

player i has m_i # of actions.

Need to find (s_1^*, s_2^*, s_3^*) s.t.

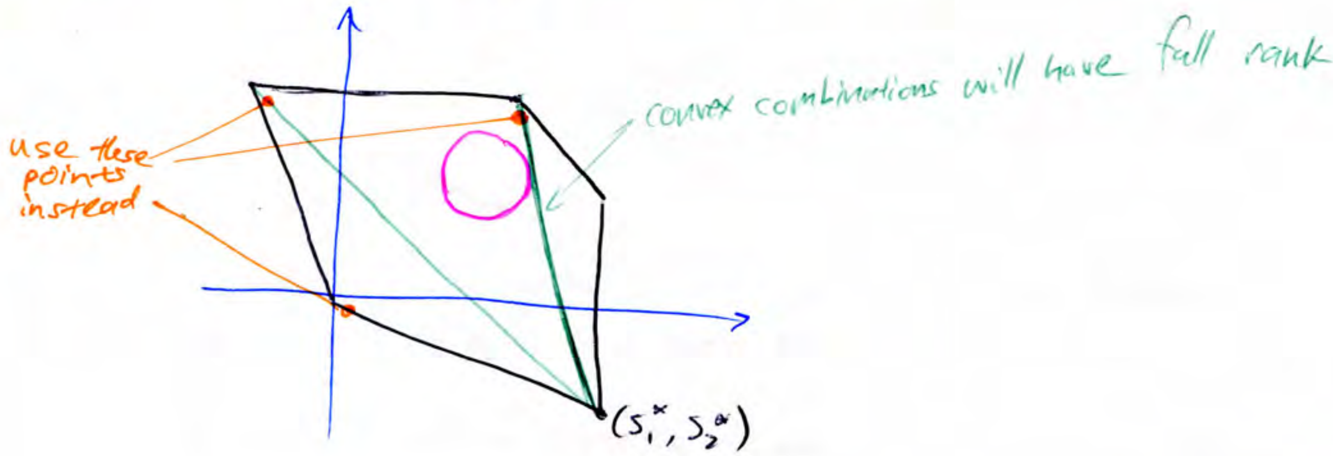
$$\begin{pmatrix} \pi(s_1^*, s_2^*, s_3^*) \\ \pi(s_1', s_2^*, s_3^*) \\ \pi(s_1^*, s_2', s_3^*) \end{pmatrix} \text{ has rank } m_1 + m_2 - 1$$

$$\begin{pmatrix} \pi(s_1^*, s_2^*, s_3^*) \\ \pi(s_1', s_2^*, s_3^*) \\ \pi(s_1^*, s_2^*, s_3') \end{pmatrix} \text{ has rank } m_1 + m_3 - 1$$

$$\begin{pmatrix} \pi(s_1^*, s_2^*, s_3^*) \\ \pi(s_1^*, s_2', s_3^*) \\ \pi(s_1^*, s_2^*, s_3') \end{pmatrix} \text{ has rank } m_2 + m_3 - 1$$

i.e. pairwise satisfies the definition.

What if the (s_1^*, s_2^*) is not efficient.



Next time: Fudenberg–Levine
Fudenberg – Kreps–Maskin } short-run + long-run players

+ Repeated games with private information

Mailith-Morris
Matushima
Ely-Horner-Olszewsky